Lecture 15. Fourier Transform techniques

(1) $$\boxed{\phi_n(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta}$$

For any function $f(\theta)$, require $\phi(\theta) \overset{\text{e.g. matches}}{=} f(\theta)$ for which:

(2) $$\int_0^{2\pi} \phi_n(\theta) \begin{matrix}\cos\\ \sin\end{matrix} n\theta \, d\theta = \int_0^{2\pi} f(\theta) \begin{matrix}\cos n\theta\\ \sin n\theta\end{matrix} \, d\theta$$

for $n = 0, 1, \ldots n$. The left side equals $\pi \begin{matrix} a_n \\ b_n \end{matrix}$, $n = 1, \ldots n$ and $2\pi a_0$ for $k = 0$. The Fourier coefficients a_k, b_k are then:

(3) $$\boxed{\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \begin{matrix}\cos\\ \sin\end{matrix} n\theta \, d\theta}, \quad n = 1, 2 \ldots n$$

orthogonality relationship

(4) $\phi(\theta) = f(\theta)$ because $\int_0^{2\pi} \cos p\theta \cos q\theta \, d\theta = \int_0^{2\pi} \cos p\theta \sin m\theta \, dx = \int_0^{2\pi} \sin p\theta \sin q\theta = 0$ where p, q, m are integer and $p \neq q$. If $p = q$ the integral $= \pi$.

Thus the left side equals $\pi \begin{pmatrix} a_n \\ b_n \end{pmatrix}$, and (2) follows.

Since $\sin n\theta = \dfrac{e^{in\theta} - e^{-in\theta}}{2i}$ and the $\sin n\theta$ term will select odd functions where $f(\theta) = -f(-\theta)$, we

Can express (1):

$$\phi_n(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n i \sin n\theta$$

$$= \sum_{n=0}^{\infty} a_n \frac{e^{i\theta n} + e^{-i\theta n}}{2} + \sum_{n=-\infty}^{-1} \underbrace{b_n}_{= -a_n} \cancel{i} \frac{e^{i\theta n} - e^{-i\theta n}}{2\cancel{i}}$$

(1b) $$\phi_n(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{i\theta n} \qquad \theta = \frac{2\pi j}{p}$$

We can let $\theta = \frac{2\pi x}{L}$, and note the equivalent of (4) is

(4b) $$\int_0^L e^{-2\pi i m x/L} e^{2\pi i n x/L} = L\delta_{mn} .$$

Thus the equivalent of (2) can be written

$$\int_0^L \phi_n(x) e^{-2\pi i n x/L} dx = \int_0^L f(x) e^{-2\pi i n x/L} dx$$

$$\sum_{n=-\infty}^{\infty} a_n \int_0^L e^{-2\pi i n x/L} e^{2\pi i m x/L} dx = \int_0^L f(x) e^{-2\pi i m x/L} dx$$

(3b) $$L a_n = \int_0^L f(x) e^{-2\pi i n x/L} dx$$

Now if we define $k = n/L$, we can express

The transforms more symmetrically:

$$\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x/L} = \int_{-\infty}^{\infty} a_n e^{2\pi i n x/L}\, dn$$

$$\shortparallel$$

$$f(x) \underset{\text{since } dk = dn/L}{=} L \int_{-\infty}^{\infty} a_n e^{2\pi i k x}\, dk$$

If we now define $L a_n = F(k)$, define the integral in (3b) between $-L/2$ and $L/2$, and let $L \to \infty$, we find

$$(5) \qquad \boxed{\begin{aligned} f(x) &= \int_{-\infty}^{\infty} F(k)\, e^{2\pi i k x}\, dk \\ F(k) &\underset{(3b)}{=} \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i k x}\, dx \end{aligned}}$$

This is a beautifully symmetric transform pair. The first equation is the inverse $(+i)$ Fourier transform. The second equation $(-i)$ is the forward Fourier transform. k is the oscillation frequency which equals $1/\lambda$ where λ is the wavelength.

If we define $k' = \frac{2\pi n}{L} = 2\pi k = 2\pi/\lambda$ we find

(6)
$$\boxed{\begin{aligned} f(x) &= \int_{-\infty}^{\infty} F(k')\, e^{ik'x}\, dk' \\ F(k') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ik'x}\, dx \end{aligned}}$$

inverse FT

forward FT

where $k' = 2\pi/\lambda$. This transform pair is more physically intuitive but involves a normality constant $1/2\pi$ that is mathematically displeasing. We'll use this transform pair, since we need physically intuitive results. Notice that we place the normality constant in the forward transform, rather than the inverse, where it would naturally appear. In either case the meaning of the transforms is clear. The forward transform gives the amplitude of cos (real) and sin (complex) wavenumbers $(2\pi/\lambda)$ that must be added together to produce $f(x)$. These amplitudes are expressed as a function $F(k)$. For some particular wavelength λ, the $\cos \frac{2\pi x}{\lambda}$ coefficient is the real part of $F(2\pi/\lambda)$, and the $\sin \frac{2\pi x}{\lambda}$ coefficient is the complex part of $F\left(\frac{2\pi}{\lambda}\right)$. Programs exist to do the fourier transforms, and these numerical transforms have been cleverly optimized so that they are very fast.

In two dimensions we define wave numbers in both the x and y directions. Call them k_x and k_y. The transform pair is then

(7)
$$\boxed{\begin{aligned} \bar{u}(k_x, k_y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y)\, e^{-ik_x x}\, e^{-ik_y y}\, dx\, dy \\ u(x,y) &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{u}(k_x, k_y)\, e^{ik_x x}\, e^{ik_y y}\, dx\, dy \end{aligned}}$$

where the exponential products might also be written

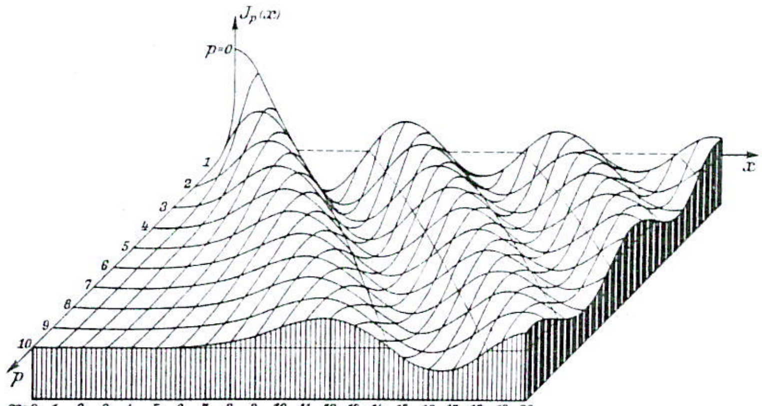
$e^{ik_x x}\, e^{ik_y y} = e^{i\underline{k}\cdot\underline{r}}$.

Often it is useful to transform in cylindrical or spherical coordinate systems. When there is perfect cylindrical symmetry, the 2D transform pair is

(8)
$$\boxed{\begin{aligned} F(k) &= \int_{r=0}^{\infty} f(r)\, J_0(kr)\, r\, dr \\ f(r) &= \int_{k=0}^{\infty} F(k)\, J_0(kr)\, k\, dk \end{aligned}}$$

where $J_0(kr)$ is a Bessel function of zero order. These functions are similar to sin or cos and can be plotted:

VIII. Zylinderfunktionen.
VIII. Bessel functions.



Jahnke + Emde, Tables of Functions
Dover, N.Y. 1945

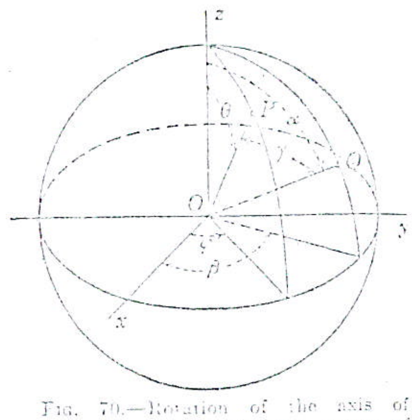


Fig. 70.—Rotation of the axis of reference from OP to OQ for a system of zonal harmonics.

Data on a sphere can be transformed if it is represented on 2B latitude circles and then sampled at 2B points around each latitude circle along lines of longitude. Around each latitude circle, ϕ runs from 0 to 2π. Along each longitude circle, θ runs from 0 to π. B is the Bandwidth of the transformation and equals half the number of subdivisions (decimations) of the latitude circles (and half-longitude circles).

$$f(\theta,\phi) = \sum_{\ell=0}^{\ell} \sum_{m=0}^{m=\ell} F(\ell,m) \quad e^{im\phi} P_\ell^m(\cos\theta)$$

(9)

$$F(\ell,m) = \int_0^{\pi} \underbrace{\left[\int_0^{2\pi} e^{-im\phi} f(\theta,\phi)\, d\phi\right]}_{\text{fourier transform}} P_\ell^m(\underset{\mu}{\underset{\shortparallel}{\cos\theta}}) \underset{d\mu}{\underset{\shortparallel}{\underbrace{\sin\theta\, d\theta}}}$$

In practice we perform the transformation on discrete data points. The points are equally spaced around circles of latitude that are also equally spaced. The ϕ locations on each latitude circle are $\phi_k = \frac{2\pi k}{2B}$; the θ locations along each half latitude circle connecting the ϕ_k on all the latitude circles from N to S pole are $\theta_j = \pi(2j+1)/4B$. This discretization replaces the integrals in (9) with sums. The fourier transform becomes:

$$\int_0^{2\pi} e^{im\phi} f(\theta,\phi)\, d\phi = \sum_{k=0}^{2B-1} e^{im\phi_k} f(\theta,\phi) \frac{2\pi}{2B}$$

and if we define $\sin\theta_j \, d\theta_j = \sin\theta_j \frac{\pi}{2B} \equiv \omega_j$, the forward transform becomes

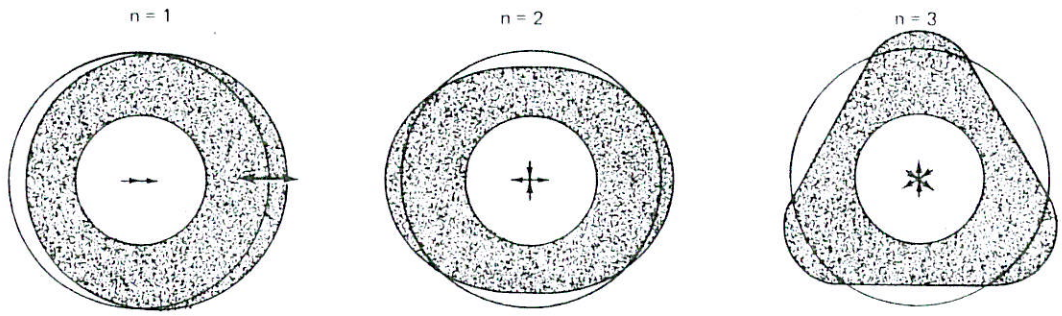
$$F(l,m) = \frac{2\pi}{2B} \sum_{j=0}^{2B-1} \sum_{k=0}^{2B-1} \omega_j f(\theta_j,\phi_k) e^{-im\phi_k} P_l^m(\cos\theta_j)$$

In fact the discretization destroys the orthonormality &

the Legendre polynomials, but this can be reclaimed by slightly modifying w_j. The resulting spherical harmonic transformations are:

$$\boxed{\begin{aligned} f(\theta,\phi) &= \sum_{l=0}^{l} \sum_{m=0}^{l} F(l,m)\, e^{im\phi}\, P_l^m(\cos\theta) \\ F(l,m) &= \frac{2\pi}{2B} \sum_{j=0}^{2B-1} \sum_{k=0}^{2B-1} w_j\, f(\theta_j,\phi_k)\, e^{-im\phi_k}\, P_l^m(\cos\theta_j) \\ \phi_k &= \frac{2\pi k}{2B}, \quad \theta_j = \pi(2j+1)/4B. \end{aligned}}$$

The transform is achieved by first discretely fourier transforming the data along each latitude circle (2B subdivisions of each circle), and then Legendre transforming each ϕ_k. Subscript l corresponds to the wave number k. As l becomes large, $k = \dfrac{l+1/2}{r}$. The Bessel functions $P_l^m(\cos\theta)$ are again roughly like cosine and sine functions



When the transforms are performed with a finite number of discretization on finite half spaces or spheres, numerical consideration arise in all cases (5, 6, and 7 as well as 8). For example:

(1) data is imaged at adjacent "mirror" squares when data on a plane is transformed in a square of finite dimension.

(2) The Bessel functions in (8) (the cylindrical symmetry case) should be integrated to their zeros to avoid spreading — then — necessary errors.