Lecture 16 Glacial Rebound Equations and Their Solution

I. The Equations

For slow deformations where momentum is unimportant, $\rho \frac{Du_i}{Dt} = 0$, and the Cauchy conservation of momentum equation (5-4 in notes) becomes:

(1) $$\rho \underline{g} + \underline{\nabla} \cdot \underline{\underline{\tau}} = 0$$

The material we consider (the earth) lies in a self-generated gravitational field and the earth is hydrostatically pre-stressed (eg pressure increases with depth). This pre-stress causes no deformation and is thus conveniently subtracted out. The stresses are applied to the earth's surface by redistributions / load on the surface and this load-redistribution causes the deformation. If we let

$$\tau_{ij} = -p_0(z)\,\delta_{ij} + \sigma_{ij}$$ ← Deviatoric stress tensor — stress tensor with hydrostatic pressure subtracted out

Then (1) becomes:

(2) $$\rho \underline{g} - \underline{\nabla} p_0 + \underline{\nabla} \cdot \underline{\underline{\sigma}}_{ij} = 0$$

From our previous discussion we know this equation must apply differently to an elastic solid and a fluid. A load floats in a fluid because the deformed fluid moves through the equilibrium (rest state) pressure field. A deformed solid carries its zero-order pressure field with it when it deforms. There is no elastic buoyancy. Love realized this in 1911. Pressure is advected in elastic deformation. If the pressure in the elastic body is p and a load is applied at t_0, a very short time, δt, later, after the elastic deformation is completed, the pressure will be

(3) $$p_0\Big|_{t+\delta t} = p_0\Big|_{t_0} - \underline{u}^e \cdot \underline{\nabla} p_0$$

where the superscript e indicates elastic displacement.

Now linearize the problem by expanding ρ and $\underline{g}$ into rest state (no deformation) and perturbation

$$\rho(\underline{x},t) = \rho_0(x_1) + \rho_1(\underline{x},t) \tag{4}$$

$$\underline{g}(\underline{x},t) = g_0(x_1) + g_1(x,t).$$

Note from (1) that when $\underline{\underline{\sigma}} = 0$, the rest state is

$$\underline{\nabla} p_0 - \rho_0 \underline{g_0} = 0 \; . \tag{5}$$

Substitution into (2) for the elastic equation where

$p_0\big|_{t+\delta t} = p_0\big|_{t_0} - \underline{u}^e \cdot \underline{\nabla} p_0$ yields

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} \underbrace{- \nabla p_0 + \underline{\nabla}(\underline{u}^e \cdot \nabla p_0) + \rho_0 \underline{g_0}}_{\underset{(4,5)}{=0}} + \rho_1 \underline{g_0} + \rho_0 \underline{g_1} =$$

The viscous equation is:

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} \underbrace{- \nabla p_0 + \rho_0 \underline{g_0}}_{\overset{\shortparallel}{0}} + \rho_1 \underline{g_0} + \rho_0 \underline{g_1} = 0$$

Then

Elastic Eqn (6) $$\underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{\nabla}(\underline{u}^e \cdot \underline{\nabla} p_0) + \rho_1 \underline{g_0} + \rho_0 \underline{g_1} = 0$$

viscous Eqn (7) $$\underline{\nabla} \cdot \underline{\underline{\sigma}} + \rho_1 \underline{g_0} + \rho_0 \underline{g_1} = 0$$

These equations can be simplified by expressing ρ_1 in another form. To first order the continuity equation is

$\frac{\partial \rho_1}{\partial t} + \underline{\nabla} \cdot \rho_0 \underline{v} = 0$. Integrating over δt we find

$$\rho_1\Big|_{t+\delta t} = \underline{\nabla} \cdot \rho_0 \underline{u}^e = \underline{u}^e \cdot \nabla \rho_0 + \rho_0 \underline{\nabla} \cdot \underline{u}_e = u_1^e \frac{\partial \rho_0}{\partial x_1} + \rho_0 \underline{\nabla} \cdot \underline{u}_e .$$

The second term in the elastic equation can be expressed:

$$\nabla\left(u^e \cdot \nabla p_0\right) = \nabla\left(u^e \rho_0 \underline{g}_0\right) = -\nabla u_1 \rho_0 g_0$$

$$= -\rho_0 \underline{\nabla} u_1 g_0 - g_0 u_1 \frac{\partial \rho_0}{\partial x_1} \hat{x}_1 .$$

Then

$$\underline{g_0} \rho_1 + \nabla(u^e \cdot \nabla p_0) = \hat{x}_1 g_0 \left(\cancel{u_1^e \frac{\partial \rho_0}{\partial x_1}} + \rho_0 \nabla \cdot u_e \right) - \rho_0 g_0 \nabla u_1^e - \cancel{g_0 u_1 \frac{\partial \rho_0}{\partial x_1}}$$

$$= \rho_0 g_0 \nabla \cdot u^e \hat{x}_1 - \rho_0 \nabla u_1^e g_0$$

Thus (6) becomes

Elastic (6a)

$$\boxed{\underline{\nabla} \cdot \underline{\underline{\sigma}} - \rho_0 \underline{\nabla} u_1^e g_0 + \rho_0 g_0 \left(\nabla \cdot u^e\right) \hat{x}_1 + \rho_0 g_1 = 0}$$

And by similar methods the viscous equation (7) is

viscous (7a)

$$\boxed{\underline{\nabla} \cdot \underline{\underline{\sigma}} + \left(\rho_0 g_0 \nabla \cdot u^e + g_0 u_1^v \frac{\partial \rho_0}{\partial x_1}\right) \hat{x}_1 + \rho_0 g_1 = 0}$$

Notice that we assume change in material density is due to elastic deformation, while buoyancy arises entirely from viscous movements through the zero-order pressure field. Remember $\hat{x}_1 = \hat{r}$ or $\hat{z}$. In fact $\frac{\partial \rho_0}{\partial x_1}$ is the non-adiabatic density gradient. If the fluid changes density and temperature so it is exactly similar to its surroundings, no buoyant forces arise. Finally, notice that pressure does not occur in these equations. We apply stresses to the surface of the earth by re-distributing loads and these stresses cause elastic and viscous deformation. Avoiding the complication of pressure, a scalar separate from the stress tensor, is a conceptual advantage.

II. Solution of the Glacial Rebound Equations

In the case of the *zonally-stratified* earth, we seek solutions where the elastic and viscous parameters vary mainly with depth. *Propagator matrices* and *Runge-Kutta integration* provide the most useful method of solution when parameter variations are in one coordinate direction only. In these methods we remove the derivatives of u_e and v_v (elastic deformation and fluid flow) by fourier-transforming the equations on horizontal (or spherical) planes. We are then left only with vertical gradients that we can easily (or elegantly) integrate. Again physics induces us to learn some nice mathematics.

A. Reduction of the Elastic equation to matrix form

The elastic constitutive relations are

$$\sigma_{ij} = \lambda \Theta \delta_{ij} + 2\mu e_{ij}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

$$\Theta = \frac{\partial u_k}{\partial x_k}$$

$$\sigma = \lambda + 2\mu.$$

If we denote the fourier transform with a bar, since

$$\bar{u}(k_x, k_y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y,z)\, e^{-ik_x x}\, e^{-ik_y y}\, dx\, dy$$

$$\partial_x \bar{u} = i k_x \bar{u}$$

Thus, if we seek the solution of the fourier transformed version of (6a) assuming g_0 = constant, $g_1 = 0$; and neglect hydrostatic pre-stress in an incompressible solid ($\nabla \cdot u = 0$)

(8) $$\underline{\nabla} \cdot \underline{\underline{\sigma}} = 0$$

In fourier transformed variable this can be expressed

The equation in transformed variables is:

$$\text{(a)} \quad ik_x \bar\sigma_{xx} + ik_y \bar\sigma_{yx} + \partial_z \bar\sigma_{zx} = 0$$

$$\text{(b)} \quad ik_x \bar\sigma_{xy} + ik_y \bar\sigma_{yy} + \partial_z \bar\sigma_{zy} = 0$$

$$\text{(c)} \quad ik_x \bar\sigma_{xz} + ik_y \bar\sigma_{yz} + \partial_z \bar\sigma_{zz} = 0$$

The constitutive relations can be transformed similarly:

$$\text{(d)} \quad \bar\sigma_{xx} = \sigma\, ik_x \bar u_x + \lambda\left(ik_y \bar u_y + \partial_z \bar u_z\right)$$

$$\text{(e)} \quad \bar\sigma_{yy} = \sigma\, ik_y \bar u_y + \lambda\left(ik_x \bar u_x + \partial_z \bar u_z\right)$$

$$\text{(f)} \quad \bar\sigma_{zz} = \sigma\, \partial_z \bar u_z + \lambda\left(ik_x \bar u_x + ik_y \bar u_y\right)$$

$$\text{(g)} \quad \bar\sigma_{xy} = \mu\left(ik_y \bar u_x + ik_x \bar u_y\right)$$

$$\text{(h)} \quad \bar\sigma_{xz} = \mu\left(\partial_z \bar u_x + ik_x \bar u_z\right)$$

$$\text{(i)} \quad \bar\sigma_{yz} = \mu\left(\partial_z \bar u_y + ik_y \bar u_z\right)$$

from

(h') Thus $\partial_z \bar u_x = -ik_x \bar u_z + \mu^{-1} \bar\sigma_{xz}$

$$\text{(i')} \quad \partial_z \bar u_y = -ik_y \bar u_z + \mu^{-1} \bar\sigma_{yz}$$

$$\text{(f')} \quad \partial_z \bar u_z = -\sigma^{-1}\lambda\left(ik_x \bar u_x + ik_y \bar u_y\right) + \sigma^{-1} \bar\sigma_{zz}$$

$$\text{(c')} \quad \partial_z \bar\sigma_{zz} = -ik_x \bar\sigma_{xz} - ik_y \bar\sigma_{yz}$$

$$\text{(a')} \quad \partial_z \bar\sigma_{xz} = -ik_x \bar\sigma_{xx} - ik_y \bar\sigma_{yx}$$

$$\text{(b')} \quad \partial_z \bar\sigma_{yz} = -ik_x \bar\sigma_{xy} - ik_y \sigma_{yy}$$

Since boundary conditions apply at $\hat{z} \cdot \underline{\underline{\sigma}}$, we need to eliminate $\bar{\sigma}_{xy}$, $\bar{\sigma}_{xx}$, $\bar{\sigma}_{yy}$ from the last two equations. Consequently

$$(a') \qquad \partial_z \sigma_{xz} = -ik_x \Big\{ \overbrace{\sigma i k_x \bar{u}_x + \lambda \Big[i k_y \bar{u}_y}^{(d)} \; \overbrace{- \sigma^{-1} \lambda (i k_x \bar{u}_x + i k_y \bar{u}_y)}^{(f')}$$

$$+ \sigma^{-1} \bar{\sigma}_{zz} \Big] \Big\} \; - ik_y \overbrace{\big(\mu (i k_y \bar{u}_x + i k_x \bar{u}_y) \big)}^{(g)}$$

$$= \bar{u}_x \left(k_x^2 \sigma - \lambda^2 \sigma^{-1} k_x^2 + k_y^2 \mu \right)$$

$$\bar{u}_y \left(\lambda k_x k_y - \sigma^{-1} \lambda^2 k_x k_y + \mu k_x k_y \right)$$

$$\bar{\sigma}_{zz} \left(-i k_x \lambda \sigma^{-1} \right)$$

$$(a'') \qquad \partial_z \sigma_{xz} = \left(4 \sigma^{-1} \mu (\lambda + \mu) k_x^2 + \mu k_y^2 \right) \bar{u}_x + \sigma^{-1} \mu (3\lambda + 2\mu) k_x k_y \bar{u}_y$$
$$- i k_x \lambda \sigma^{-1} \bar{\sigma}_{zz}$$

$$(b') \qquad \partial_z \sigma_{yz} = -ik_x \Big[\overbrace{\mu (i k_y \bar{u}_x + i k_x \bar{u}_y)}^{(g)} \Big] - ik_y \Big[\overbrace{\sigma i k_y \bar{u}_y}^{(e)}$$

$$+ \lambda \Big\{ i k_x \bar{u}_x - \overbrace{\sigma^{-1} \lambda (i k_x \bar{u}_x + i k_y \bar{u}_y)}^{(f')} + \sigma^{-1} \bar{\sigma}_{zz} \Big\} \Big].$$

$$= \bar{u}_x \left(\mu k_x k_y + \lambda k_x k_y - \lambda^2 \sigma^{-1} k_x k_y \right)$$

$$+ \bar{u}_y \left(k_x^2 \mu + \sigma k_y^2 - \lambda^2 \sigma^{-1} k_y^2 \right)$$

$$- i k_y \lambda \sigma^{-1} \bar{\sigma}_{zz}$$

$$(b'') \quad \partial_z \sigma_{yz} = \sigma^{-1}\mu(3\lambda+2\mu)k_x k_y \bar{u}_x + \left(4\sigma^{-1}\mu(\lambda+\mu)k_y^2 + \mu k_x^2\right)\bar{u}_y - ik_y\sigma^{-1}\lambda\,\bar{\sigma}_{zz}$$

Equations h', i', f', a'', b'', c' can be written in matrix form:

$$\partial_z \begin{bmatrix} \bar{u}_x \\ \bar{u}_y \\ \bar{u}_z \\ \bar{\sigma}_{xz} \\ \bar{\sigma}_{yz} \\ \bar{\sigma}_{zz} \end{bmatrix} = \begin{array}{c} \begin{array}{cccccc} \bar{u}_x & \bar{u}_y & \bar{u}_z & \bar{\sigma}_{xz} & \bar{\sigma}_{yz} & \bar{\sigma}_{zz} \end{array} \\ \begin{bmatrix} 0 & 0 & -ik_x & \mu^{-1} & 0 & 0 \\ 0 & 0 & -ik_y & 0 & \mu^{-1} & 0 \\ -\sigma^{-1}\lambda ik_x & -\sigma^{-1}\lambda ik_y & 0 & 0 & 0 & \sigma^{-1} \\ 4\sigma^{-1}\mu(\lambda+\mu)k_x^2 + \mu k_y^2 & \sigma^{-1}\mu(3\lambda+2\mu)k_x k_y & 0 & 0 & 0 & -\lambda\sigma^{-1}ik_x \\ \sigma^{-1}\mu(3\lambda+2\mu)k_x k_y & 4\sigma^{-1}\mu(\lambda+\mu)k_y^2 + \mu k_x^2 & 0 & 0 & 0 & -\lambda\sigma^{-1}ik_y \\ 0 & 0 & 0 & -ik_x & -ik_y & 0 \end{bmatrix} \end{array} \begin{bmatrix} \bar{u}_x \\ \bar{u}_y \\ \bar{u}_z \\ \bar{\sigma}_{xz} \\ \bar{\sigma}_{yz} \\ \bar{\sigma}_{zz} \end{bmatrix}$$

This set of six coupled, first order, ordinary differential equations separates into a set of 2 and a set of 4. To see this let $k_y \to 0$

$k_y \to 0$

$$\partial_z \begin{bmatrix} \bar{u}_x \\ \bar{u}_y \\ \bar{u}_z \\ \bar{\sigma}_{xz} \\ \bar{\sigma}_{yz} \\ \bar{\sigma}_{zz} \end{bmatrix} = \begin{array}{cccccc} \bar{u}_x & \bar{u}_y & \bar{u}_z & \bar{\sigma}_{xz} & \bar{\sigma}_{yz} & \bar{\sigma}_{zz} \\ \hline 0 & 0 & -ik_x & \mu^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu^{-1} & 0 \\ -\sigma^{-1}\lambda i k_x & 0 & 0 & 0 & 0 & \sigma^{-1} \\ 4\sigma^{-1}\mu(\lambda+\mu)k_x^2 & 0 & 0 & 0 & 0 & -\lambda\sigma^{-1}ik_x \\ 0 & \mu k_x^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -ik_x & 0 & 0 \end{array} \begin{bmatrix} \bar{u}_x \\ \bar{u}_y \\ \bar{u}_z \\ \bar{\sigma}_{xz} \\ \bar{\sigma}_{yz} \\ \bar{\sigma}_{zz} \end{bmatrix}$$

Write in separate form:

(9) non-divergent elastic

$$\partial_z \begin{bmatrix} \bar{u}_y \\ \bar{\sigma}_{yz} \end{bmatrix} = \begin{bmatrix} 0 & \mu^{-1} \\ \mu k_x^2 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_y \\ \bar{\sigma}_{yz} \end{bmatrix}$$

(10) Divergent elastic

$$\partial_z \begin{bmatrix} \bar{u}_x \\ \bar{u}_z \\ \bar{\sigma}_{xz} \\ \bar{\sigma}_{zz} \end{bmatrix} = \begin{bmatrix} 0 & -ik_x & \mu^{-1} & 0 \\ -\sigma^{-1}\lambda i k_x & 0 & 0 & \sigma^{-1} \\ 4\sigma^{-1}\mu(\lambda+\mu)k_x^2 & 0 & 0 & -\lambda\sigma^{-1} i k_x \\ 0 & 0 & -ik_x & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_x \\ \bar{u}_z \\ \bar{\sigma}_{xz} \\ \bar{\sigma}_{zz} \end{bmatrix}$$

It is easy to see the same result would have been obtained had we let $k_x \to 0$, except the 2×2 system would be $\bar{u}_x, \bar{\sigma}_{xz}$, and the 4×4 $\bar{u}_y, \bar{u}_z, \bar{\sigma}_{yz}, \bar{\sigma}_{zz}$. Thus (9) and (10) are valid for any wave number $\mathbf{k}$, regardless of its orientation.

Physically (9) describes non-divergent deformation on surfaces perpendicular to $\hat{z}$, and (10) describes divergent flow on surfaces $\perp$ to $\hat{z}$. Motion lacking divergence or convergence on a surface $\perp$ to $\hat{z}$ can never produce flow $\perp$ to $\hat{z}$ and must remain parallel to the surface. Similarly a motion which has a component $\perp$ to $\hat{z}$ can never produce a non-divergent flow on such a surface. Thus the two flows are mutually independent. In fact if $\bar{\sigma}_{yz}$ and $\bar{u}_y = 0$ on any surface in the body they are zero on all.. For example if $\bar{\sigma}_{yz} = u_y = 0$ at some surface, from (9) $\partial_z \bar{u}_y = \partial_z \bar{\sigma}_{yz} = 0$ at that surface, and $\bar{\sigma}_{xz}$ and $\bar{u}_y$ must be zero on the adjacent surfaces, and on all surfaces in the body.

The incompressible elastic solution follows by letting $\lambda \to \infty$. Remember $\sigma = \lambda + 2\mu$ so if $\lambda \to \infty$, $\lambda\sigma^{-1} \to 1$, etc. With $\lambda \to \infty$, (10) becomes:

(11) Incompressible elastic

$$\partial_z \begin{bmatrix} \bar{u}_x \\ \bar{u}_z \\ \bar{\sigma}_{xz} \\ \bar{\sigma}_{zz} \end{bmatrix} = \begin{bmatrix} 0 & -ik_x & \mu^{-1} & 0 \\ -ik_x & 0 & 0 & 0 \\ 4\mu k_x^2 & \rho_0 g_0 ik_x & 0 & -ik_x \\ \rho_0 g_0 ik_x & 0 & -ik_x & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_x \\ \bar{u}_z \\ \bar{\sigma}_{xz} \\ \bar{\sigma}_{zz} \end{bmatrix}$$

The two $\rho_0 g_0 ik_x$ terms are the advection of the pressure field. These are not present in the fluid flow equation. As one might therefore guess the 4x4 propagator solution for an incompressible fluid corresponding to

$$\underline{\nabla}\cdot\underline{\underline{\sigma}} + g u_z \partial_z \rho_0 \hat{z} = 0$$

is:

(12) Incompressible divergent viscous equation.

$$\partial_z \begin{bmatrix} \bar{v}_x \\ \bar{v}_z \\ \bar{\sigma}_{xz} \\ \bar{\sigma}_{zz} \end{bmatrix} = \begin{bmatrix} 0 & -ik_x & \mu^{-1} & 0 \\ -ik_x & 0 & 0 & 0 \\ 4\mu k_x^2 & 0 & 0 & -ik_x \\ 0 & 0 & -ik_x & 0 \end{bmatrix} \begin{bmatrix} \bar{v}_x \\ \bar{v}_z \\ \bar{\sigma}_{xz} \\ \bar{\sigma}_{zz} \end{bmatrix} + \bar{u}_z^{\nu} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -g \partial_z \rho_0 \end{bmatrix}$$

where $\bar{u}_z = \int_0^t \bar{v}_z \, dt$.

~~Finally,~~ Finally let us simplify (11) and (12) by letting $\tilde{\mu} = \frac{\mu}{\mu^*}$, where μ^* is the rigidity of the lowest layer in our model. Then (11) can be written:

(11a)
$$\partial_z \begin{bmatrix} 2\mu^* i k \bar{u}_x \\ 2\mu^* k \bar{u}_z \\ i \bar{\sigma}_{xz} \\ \bar{\sigma}_{zz} \end{bmatrix} = k \begin{bmatrix} 0 & 1 & 2\tilde{\mu}^{-1} & 0 \\ -1 & 0 & 0 & 0 \\ 2\tilde{\mu} & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2\mu^* i k \bar{u}_x \\ 2\mu^* k \bar{u}_z \\ i \bar{\tau}_{xz} \\ \bar{\tau}_{zz} \end{bmatrix}$$

If we let the viscosity be η and define $\tilde{\eta} = \eta/\eta^*$, (12) is:

$$\partial_z \begin{bmatrix} 2\eta^* i k \bar{v}_x \\ 2\eta^* k \bar{v}_z \\ i \bar{\tau}_{xz} \\ \bar{\tau}_{zz} \end{bmatrix} = k \begin{bmatrix} 0 & 1 & 2\tilde{\eta}^{-1} & 0 \\ -1 & 0 & 0 & 0 \\ 2\tilde{\eta} & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2\eta^* i k \bar{v}_x \\ 2\eta^* k \bar{v}_z \\ i \bar{\tau}_{xz} \\ \bar{\tau}_{zz} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -g_0 \partial_z \rho_0 \bar{u}_z \end{bmatrix}$$

The issue now is how to solve these equations. Note they make great sense because the matrices are real numbers, and the important parts of the transform components of motion in the x direction are imaginary, while the important parts of the z components of motion are the real ones.

B. Propagator Solution to the Matrix Equations

Notice that the form of (11a), and (12a) provided the fluid is adiabatic (constant density) is

(13) $$\partial_z \underline{u} = \underline{\underline{A}}(z)\, \underline{u}$$

If A were a single z-dependent variable we would find the solution:

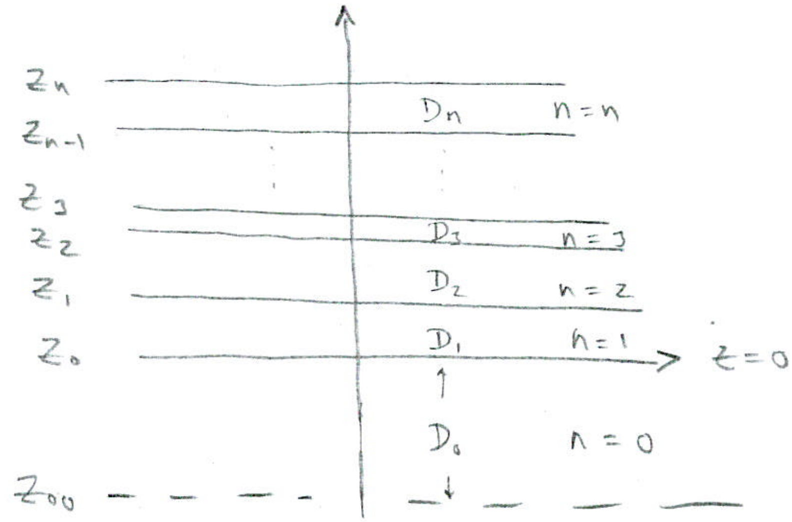
$$\int_{u(z_0)}^{u(z)} \frac{\partial u}{u} = \int_{z_0}^{z} A(z)\, dz$$

$$u(z) = u(z_0)\, e^{\int_{z_0}^{z} A(z)\, dz}$$

In fact, the same holds for our matrix equation.

(14) $$\underline{u}(z) = e^{\int_{z_0}^{z} \underline{\underline{A}}(z)\, dz}\, \underline{u}(z_0)$$

If we consider a layered half space in which $\tilde{\mu}$ or $\tilde{\eta}$ is constant in each layer, we can write (14):



(15) $$\bar{u}(z) = e^{A(z^n_{n-1})D_n}\, e^{A(z^{n-1}_{n-2})D_{n-1}} \cdots e^{A(z^1_0)D_1}\, e^{A(z^0_{00})z_{00}}\, u(z_{00})$$

In each of these terms $A(z^n_{n-1})$ is simply the matrix in (11a) or (12a) with the $\tilde{\mu}$ or $\bar{\eta}$ value appropriate for that layer. Each term in (15) is a matrix whose function is to propagate the solution from the base to the top of the layer involved. These "propagator" matrices can be designated:

(16) $$P_n = e^{A(z^n_{n-1})D_n}$$

The matrices we consider here (11a, 12a) have 4 real eigenvalues. Two equal k, and two equal −k.

This can be shown by setting $\det\left(A - \alpha I\right) = 0$.

If you do this you will find $\alpha_1 = \alpha_2 = k$, $\alpha_3 = \alpha_4 = -k$.

For such a matrix

(17) $$\underline{\underline{P}} = \underline{\underline{G}}^{+} + \underline{\underline{G}}^{-}$$

(18) $$\underline{\underline{G}}^{+} = \frac{e^{kD}}{4k^2}\left[\left\{\left(\underline{\underline{A}} + k\underline{\underline{I}}\right)^2 - \frac{1}{k}\left(\underline{\underline{A}} - k\underline{\underline{I}}\right)\left(\underline{\underline{A}} + k\underline{\underline{I}}\right)^2\right\} + D\left(\underline{\underline{A}} - k\underline{\underline{I}}\right)\left(\underline{\underline{A}} + k\underline{\underline{I}}\right)^2\right]$$

$$\underline{\underline{G}}^{-} = \frac{e^{-kD}}{4k^2}\left[\left\{\left(\underline{\underline{A}} - k\underline{\underline{I}}\right)^2 + \frac{1}{k}\left(\underline{\underline{A}} + k\underline{\underline{I}}\right)\left(\underline{\underline{A}} - k\underline{\underline{I}}\right)^2\right\} + D\left(\underline{\underline{A}} + k\underline{\underline{I}}\right)\left(\underline{\underline{A}} - k\underline{\underline{I}}\right)^2\right]$$

for our case

(19) $$G^{+} = \frac{e^{kD}}{2}\left[\begin{pmatrix} 1 & 0 & \tilde{\mu}^{-1} & 0 \\ 0 & 1 & 0 & \tilde{\mu}^{-1} \\ \tilde{\mu} & 0 & 1 & 0 \\ 0 & \tilde{\mu} & 0 & 1 \end{pmatrix} + kD\begin{pmatrix} 1 & 1 & \tilde{\mu}^{-1} & \tilde{\mu}^{-1} \\ -1 & -1 & -\tilde{\mu}^{-1} & -\tilde{\mu}^{-1} \\ \tilde{\mu} & \tilde{\mu} & 1 & 1 \\ -\tilde{\mu} & -\tilde{\mu} & -1 & -1 \end{pmatrix}\right]$$

$$G^{-} = \frac{e^{-kD}}{2}\left[\begin{pmatrix} 1 & 0 & -\tilde{\mu}^{-1} & 0 \\ 0 & 1 & 0 & -\tilde{\mu}^{-1} \\ -\tilde{\mu} & 0 & 1 & 0 \\ 0 & -\tilde{\mu} & 0 & 1 \end{pmatrix} + kD\begin{pmatrix} -1 & 1 & \tilde{\mu}^{-1} & -\tilde{\mu}^{-1} \\ -1 & 1 & \tilde{\mu}^{-1} & -\tilde{\mu}^{-1} \\ \tilde{\mu} & -\tilde{\mu} & -1 & 1 \\ \tilde{\mu} & -\tilde{\mu} & -1 & 1 \end{pmatrix}\right]$$

Remembering that $\sinh x \equiv \frac{1}{2}(e^{x} - e^{-x})$ and

$\cosh \equiv \frac{1}{2}(e^{x} + e^{-x})$, if we define

$$S = kD \sinh kD$$

$$C = kD \cosh kD$$

(20)

$$CP = \cosh kD + kD \sinh kD$$
$$CM = \cosh kD - kD \sinh kD$$

$$SP = \sinh kD + kD \cosh kD$$
$$SM = \sinh kD - kD \cosh kD$$

Then

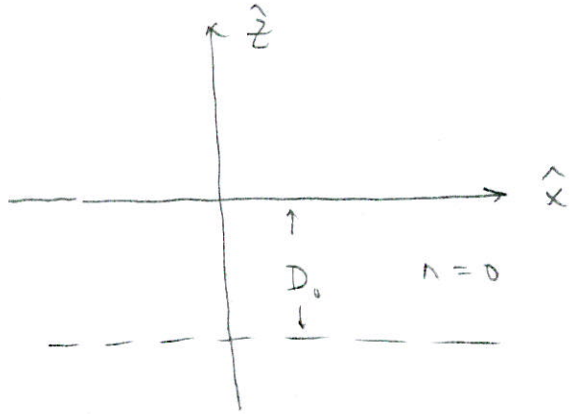
(21)
$$\underline{\underline{P}} = \begin{pmatrix} CP & C & \tilde{\mu}^{-1} SP & \tilde{\mu}^{-1} S \\ -C & CM & -\tilde{\mu}^{-1} S & \tilde{\mu}^{-1} SM \\ \tilde{\mu} SP & \tilde{\mu} S & CP & C \\ -\tilde{\mu} S & \tilde{\mu} SM & -C & CM \end{pmatrix}.$$

We can now solve any number of single problems with great ease.

C. Some Solution Using The Propagator Matrix

1. Isoviscous half space with harmonic load applied to surface

This case involves no layers.



$$\bar{u}(z) = e^{\underline{\underline{A}}(\mathring{z}_{00}) z_{00}} \, \underline{u}(z_{00})$$

Suppose the solution at the surface is $u(z=0) = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}$.

If we propagate this solution down a distance D_0,

where D_0 can be large, D will be a negative number, and for the solution to remain finite we must have $G_0^{-} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0$. Since in

Expand

the base layer $\tilde{\mu} = 1$,

This means $A = C$, $B = D$. At the

surface are two conditions on stress (vertical and horizontal) (21)

require
$$\begin{pmatrix} A \\ B \\ A \\ B \end{pmatrix} = \begin{pmatrix} - \\ - \\ 0 \\ -1 \end{pmatrix}, \text{ so } A = 0,\ B = -1.$$

Thus in general the solution $u(z) = G^{+} u(0)$

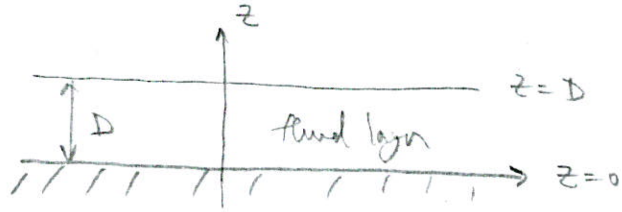
and $u(0) = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}$. Thus

(22) Homogeneous half space solution

$$\boxed{\begin{bmatrix} 2\eta^{*} i k \bar{v}_x \\ 2\eta^{*} k \bar{v}_z \\ i\,\bar{\tau}_{xz} \\ \bar{\tau}_{zz} \end{bmatrix} = e^{kz}\left[\begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix} + kz \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}\right]}$$

2. Flow confined to a layer

Instead of an isoviscous half-space, suppose flow were confined to a channel.



At $z=0$, $\bar{v}_z = \bar{v}_x = 0$, and at $z = D$ $\bar{\tau}_{xz} = 0$, $\bar{\tau}_{zz} = -1$.

Take $\tilde{\mu} = 1$ in the layer.

Then

$$\bar{u}(D) = P(D)\begin{pmatrix} 0 \\ 0 \\ C' \\ D' \end{pmatrix} = \begin{pmatrix} \sim \\ \sim \\ 0 \\ -1 \end{pmatrix}$$

or

$$\begin{pmatrix} CP & C \\ -C & Cm \end{pmatrix}\begin{pmatrix} C' \\ D' \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

The inverse of the matrix is $\left(\dfrac{\text{cofactor of } A_{ij}}{\det \underline{A}}\right)^T$, or

$$A^{-1} = \frac{1}{\det}\begin{pmatrix} Cm & -C \\ C & CP \end{pmatrix}, \quad \det = (CP)(Cm) + C^2 = C_0^2 + k^2D^2$$

Then

$$\begin{pmatrix} C' \\ D' \end{pmatrix} = \frac{1}{\det}\begin{pmatrix} C \\ -CP \end{pmatrix} = \begin{matrix} \dfrac{C}{C_0^2 + k^2D^2} \\ \dfrac{-CP}{C_0^2 + k^2D^2} \end{matrix}$$

and

$$\bar{u}(D) = \begin{bmatrix} 2\eta^* ik\bar{v}_x \\ 2\eta^* k\bar{v}_t \\ i\bar{\tau}_{xt} \\ \bar{\tau}_{zt} \end{bmatrix} = \begin{bmatrix} CP & C & SP & S \\ -C & Cm & -S & Sm \\ SP & S & CP & C \\ -S & Sm & -C & Cm \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dfrac{C}{C_0^2+k^2D^2} \\ \dfrac{-CP}{C_0^2+k^2D^2} \end{bmatrix}$$

(23) Channel flow solution

$$\boxed{\begin{bmatrix} 2\eta^* ik\bar{v}_x \\ 2\eta^* k\bar{v}_t \\ i\bar{\tau}_{xt} \\ \bar{\tau}_{zt} \end{bmatrix} = \begin{bmatrix} \dfrac{k^2D^2}{C_0^2+kD^2} \\ \dfrac{kD - C_0S_0}{C_0^2+k^2D^2} \\ 0 \\ -1 \end{bmatrix}}, \quad \text{where } \begin{matrix} C_0 = \cosh kD \\ S_0 = \sinh kD \end{matrix}$$

Now as $kD = \frac{2\pi D}{\lambda} \rightarrow$ large (short wavelength)

the $u(D) \rightarrow \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}$ which is the same as the half space

solution (22). As $kD \rightarrow$ small $(D \ll \lambda)$, the

denominator of the $2\eta^* k \bar{u}_z$ term approaches 1, while

the numerator approaches $\frac{2}{3}(kD)^3$, e.g.:

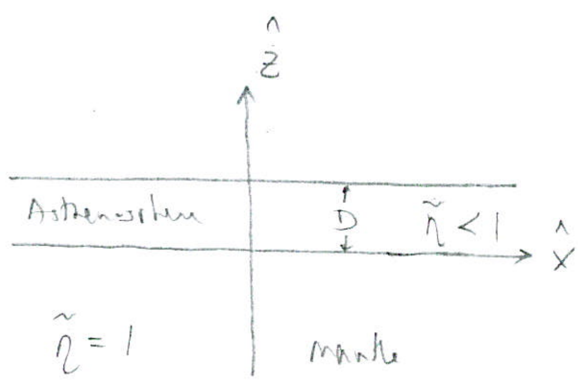
$$kD - C_0 S_1 = kD - \left(1 + \frac{(kD)^2}{2!} + \dots\right)\left(kD + \frac{(kD)^3}{3!} + \dots\right) \rightarrow \frac{2}{3}(kD)^3$$

3. Asthenosphere and Mantle

A case very relevant to the earth is

a rotating thin fluid layer overlying a less fluid

mantle:



The boundary conditions are that the solution remains finite at

large depth, which means as in §1. above, $u(z) = \begin{pmatrix} A \\ B \\ A \\ B \end{pmatrix}$, and no shear stress and unit normal load at the surface $(z=D)$:

$$\begin{pmatrix} \sim \\ \sim \\ 0 \\ -1 \end{pmatrix} = P(D) \begin{pmatrix} A \\ B \\ A \\ B \end{pmatrix} = \begin{pmatrix} CP & C & \tilde{\eta}^{-1}SP & \tilde{\eta}^{-1}S \\ -C & Cm & -\tilde{\eta}^{-1}S & \tilde{\eta}^{-1}Sm \\ \tilde{\eta}SP & \tilde{\eta}S & CP & C \\ -\tilde{\eta}S & \tilde{\eta}Sm & -C & Cm \end{pmatrix} \begin{pmatrix} A \\ B \\ A \\ B \end{pmatrix}$$

Thus

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \tilde{\eta}SP + CP & \tilde{\eta}S + C \\ -\tilde{\eta}S - C & \tilde{\eta}Sm + Cm \end{pmatrix} \begin{pmatrix} A \\ D \end{pmatrix}$$

which can be solved for A and D. It can then be determined from the preceding equation that

Asthenosphere (24) Solution

$$\boxed{2\eta^* k \bar{u}_z = \frac{\left(\tilde{\eta} + \tilde{\eta}^{-1}\right) S_0 C_0 + kD\left(\tilde{\eta} + \tilde{\eta}^{-1}\right) + \left(S_0^2 + C_0^2\right)}{2S_0 C_0 \tilde{\eta} + \left(1-\tilde{\eta}\right)^2 k^2 D^2 + \left(\tilde{\eta}^2 S_0^2 + C_0^2\right)}}$$

where $S_0 = \sinh kD$, $C_0 = \cosh kD$.

4. Lithosphere Filter

Finally consider an elastic layer overlying an inviscid fluid. The boundary conditions are that no shear stress is applied at the surface, and at the base of the layer the vertical displacement is $\bar{u}_z^B$, the vertical stress $\rho g \bar{u}_z^D$, and

The shear stress zero. Then

$$\bar{u}(\text{top}) = \begin{pmatrix} A \\ B \\ 0 \\ C' \end{pmatrix}$$

$$\bar{u}(\text{base}) = \begin{pmatrix} 2\tilde{\mu}^* k \bar{u}_z^B \\ 0 \\ \rho g \bar{u}_z^B \end{pmatrix} = P(-H) \begin{pmatrix} A \\ B \\ 0 \\ C' \end{pmatrix}$$

Then

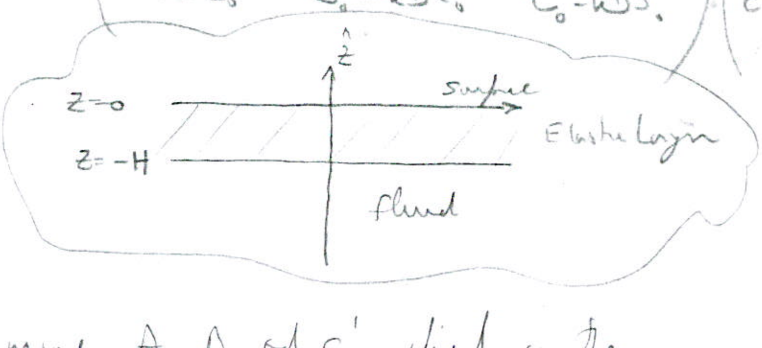
$$\begin{pmatrix} 2\mu^* k \bar{u}_z^B \\ 0 \\ \rho g u_z^B \end{pmatrix} = \begin{pmatrix} -C & Cm & Sm \\ SP & S & C \\ -S & -C & Cm \end{pmatrix} \begin{pmatrix} A \\ B \\ C' \end{pmatrix}$$

$$= \begin{pmatrix} -kDC_0 & C_0 - kDS_0 & S_0 - kDC_0 \\ S_0 + kDC_0 & kDS_0 & kDC_0 \\ -kDS_0 & S_0 - kDC_0 & C_0 - kDS_0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C' \end{pmatrix}$$

where $D = -H$

$C_0 = \cosh kD$

$S_0 = \sinh kD$



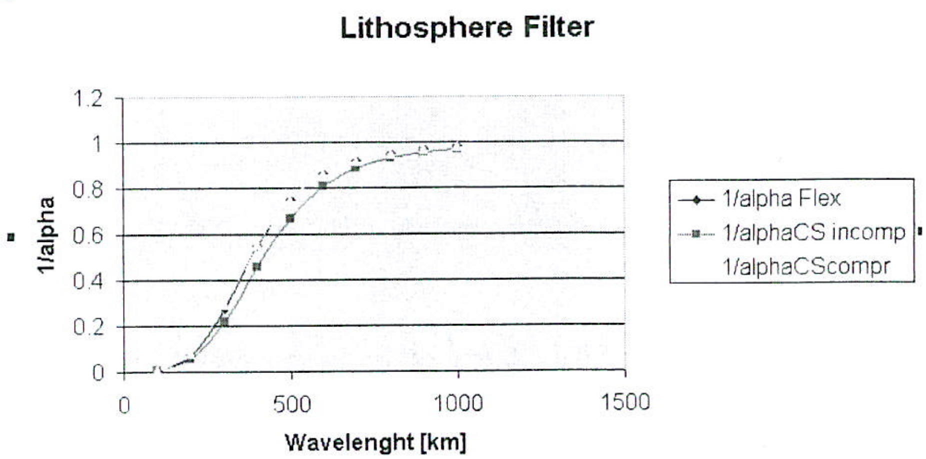
From which we can determine A, B, and C', which is the deformation at the top surface. In particular we find the ratio of the vertical strain at the surface to that at the base of the elastic layer, depend on α:

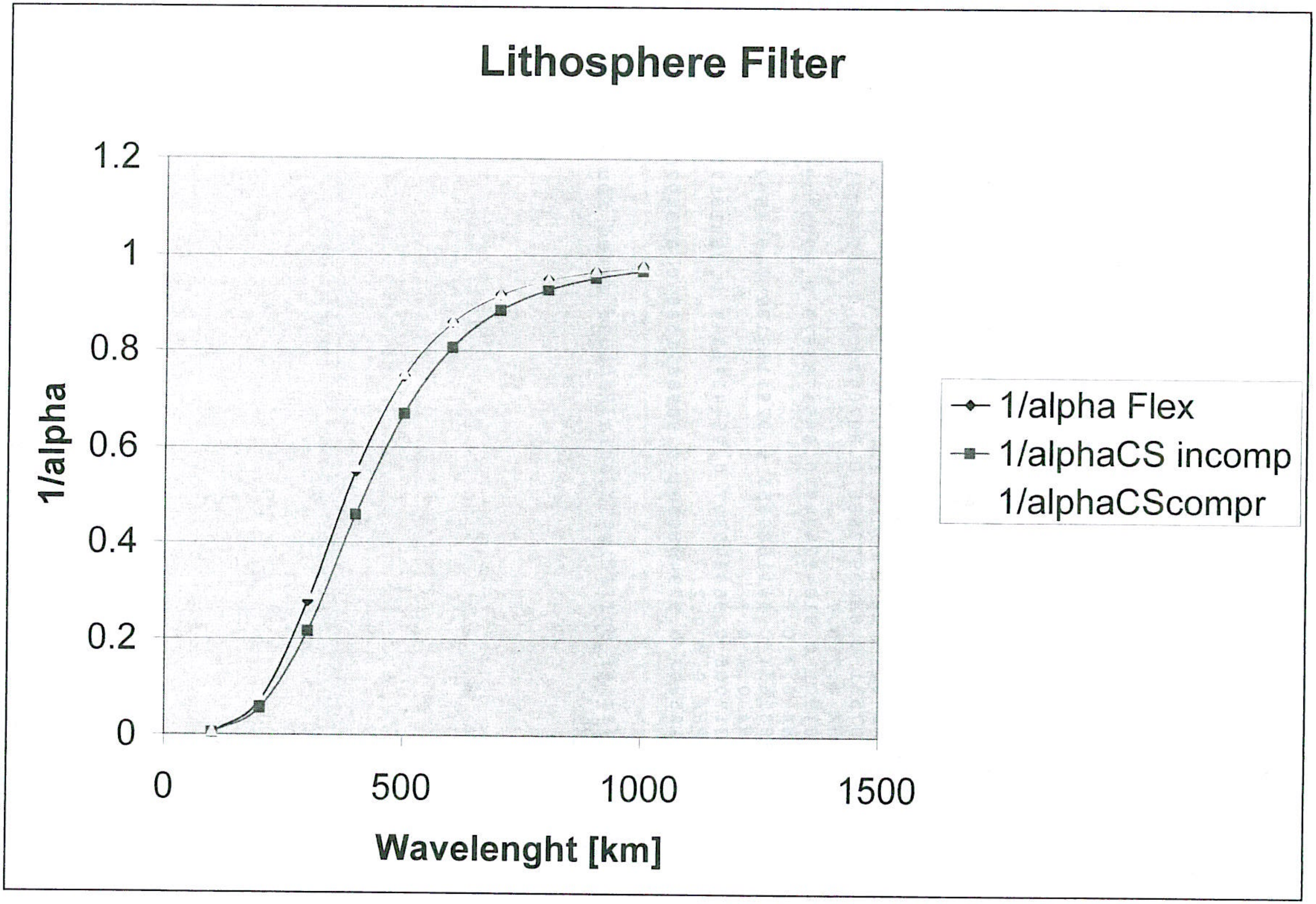
(25a) $$\boxed{\alpha = \frac{\bar{\tau}^{T}_{zz}}{\bar{u}^{B}_{z}\,\rho g} = \frac{\frac{2\mu^{*}k}{\rho g}\left(S_0^2 - k^2H^2\right) + \left(C_0 S_0 + kH\right)}{S_0 + kHC_0}}$$

It turns out that α may be alternatively derived in terms of the flexural rigidity of the elastic layer, D:

(25b) $$\boxed{\alpha = 1 + \frac{k^4 D}{\rho g}, \quad D = \frac{EH^3}{12(1-\nu^2)}},$$

where E is young's modulus $= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$, and ν is poisson's ratio $= \frac{\lambda}{2(\lambda+\mu)}$. For $H = 30$ km, $\lambda = \mu = 0.7\times10^{11}$ Pa.




Lithosphere Filter
1.2
1
0.8
0.6
0.4
0.2
0
1/alpha
0
500
1000
1500
Wavelenght [km]
1/alpha Flex
1/alphaCS incomp
1/alphaCScompr

```
Workspace C:\s2kit\s2kit10\Glacial_Uplift144\GlacialRebound

Dyalog APL/W Version 10.0.1
Serial No : 001274 / Pentium
Thu Mar 30 10:22:42 2006
      1 300 30 ALPHA 0.7 0.7
4.2E23 0.00002094395102 0.2780829232 0.2164039563 0.2927399781


      1 100 30 ALPHA 0.7 0.7
4.2E23 0.00006283185307 0.004733055167 0.004948673839 0.007398665481
      1 200 30 ALPHA 0.7 0.7
4.2E23 0.00003141592654 0.0707088494 0.05588277366 0.08148342847
      1 300 30 ALPHA 0.7 0.7
4.2E23 0.00002094395102 0.2780829232 0.2164039563 0.2927399781
      1 400 30 ALPHA 0.7 0.7
4.2E23 0.00001570796327 0.5490263102 0.458230683 0.5590150163
      1 500 30 ALPHA 0.7 0.7
4.2E23 0.00001256637061 0.7482520634 0.6703985776 0.7529896709
      1 600 30 ALPHA 0.7 0.7
4.2E23 0.00001047197551 0.8603977886 0.807042509 0.8624203129
      1 700 30 ALPHA 0.7 0.7
4.2E23 0.00000897597901 0.9194724349 0.8851783679 0.9203448905
      1 800 30 ALPHA 0.7 0.7
4.2E23 0.000007853981634 0.9511689855 0.9291169132 0.9515627159
      1 900 30 ALPHA 0.7 0.7
4.2E23 0.000006981317008 0.968945307 0.9544376344 0.9691317752
      1 1000 30 ALPHA 0.7 0.7
4.2E23 0.000006283185307 0.9794050681 0.9695821585 0.979497019

      )save
C:\s2kit\s2kit10\Glacial_Uplift144\GlacialRebound saved Thu Mar 30
...     10:23:49 2006
```