Lecture 5: Kinematics – Physical Descriptions

Chapt 3 #1-7, 10, 13, 14

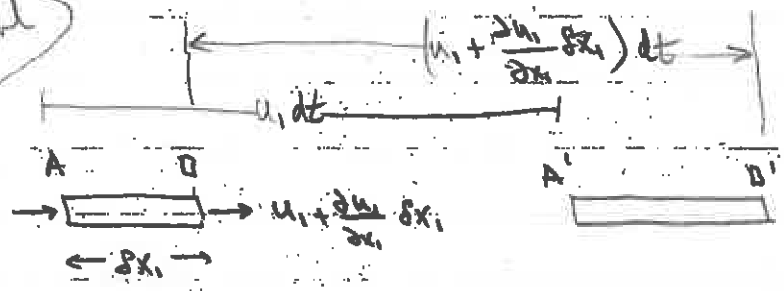
VII. Kinematic Description

Describe how a fluid moves – kinematic descr.

1. Simple examples
2. General meth.
3. Some illustrative examples
4. Tangent conv. Laws

use Taylor expansion – $u_i \to u_i + \frac{\partial u_i}{\partial x_j}\delta x_j$ small portion of the fluid

A. Linear Strain Rate



follow parcel fluid

$$\frac{1}{\delta x_1}\frac{D}{Dt}\delta x_1 = \frac{1}{dt}\,\frac{A'B' - AB}{AB} = \frac{\left(\delta x_1 + \frac{\partial u_1}{\partial x_1}\delta x_1\, dt\right) - \delta x_1}{\delta x_1}$$

linear strain rate

$$= \frac{\partial u_1}{\partial x_1}\,dt$$

B. Bulk Strain Rate

$$\frac{1}{\delta V}\frac{D}{Dt}(\delta V) = \frac{1}{\delta x_1\delta x_2\delta x_3}\frac{D}{Dt}(\delta x_1\delta x_2\delta x_3)$$

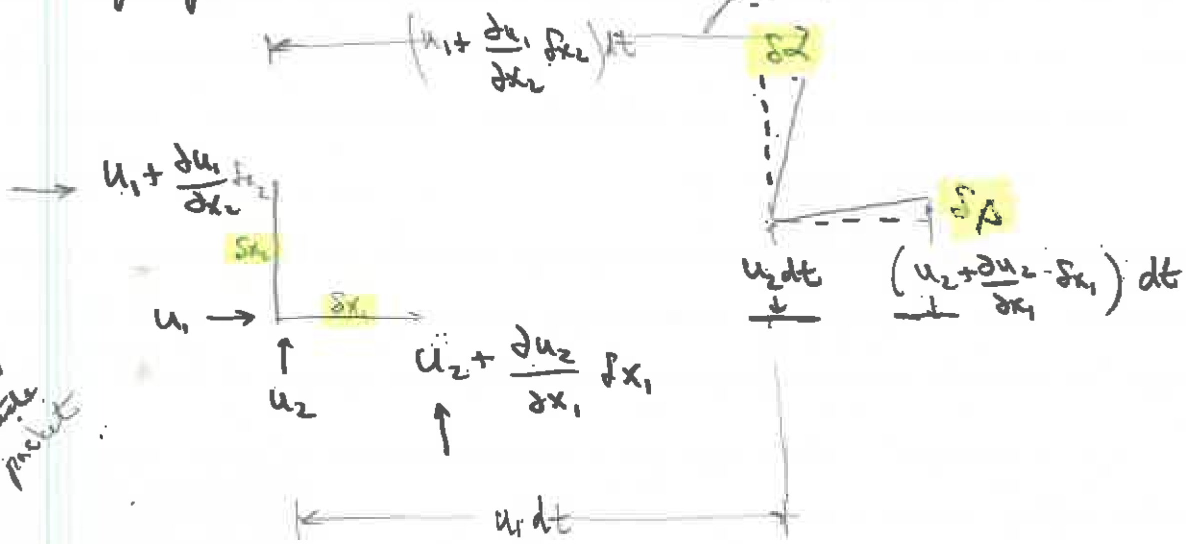
$$= \frac{\partial u_i}{\partial x_i}$$

which is the sum of the diagonal terms in the velocity gradient tensor $\frac{\partial u_i}{\partial x_j}$, and is an invariant with respect to rotation of the coordinate axes.

Deformation changes with elevation

B. Shear Strain Rate

Defined strain rate as rate change of the angle between two mutually perpendicular lines on the "fluid element."



Follow particle packet

$$\frac{d\alpha + d\beta}{dt} = \frac{1}{dt}\left\{ \frac{1}{\delta x_2}\left(\frac{\partial u_1}{\partial x_2}\delta x_2\, dt\right) + \frac{1}{\delta x_1}\left(\frac{\partial u_2}{\partial x_1}\delta x_1\, dt\right)\right\}$$

$$\therefore \text{ strain rate } = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

Combining linear strain and shear strain, we

define a strain rate tensor: Note off-diagonal terms are defined as half the strain rate

Strain Rate tensor

$$\boxed{\epsilon_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \quad (4.2)$$

The diagonal terms are the bulk strain

Symmetric matrix!

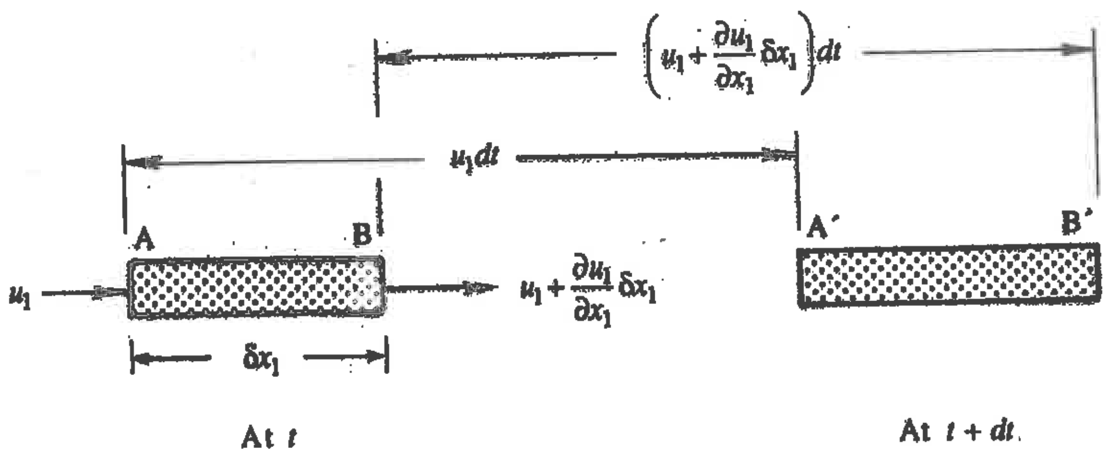


Figure 3.9 Linear strain rate. Here, $A'B' = AB + BB' - AA'$.

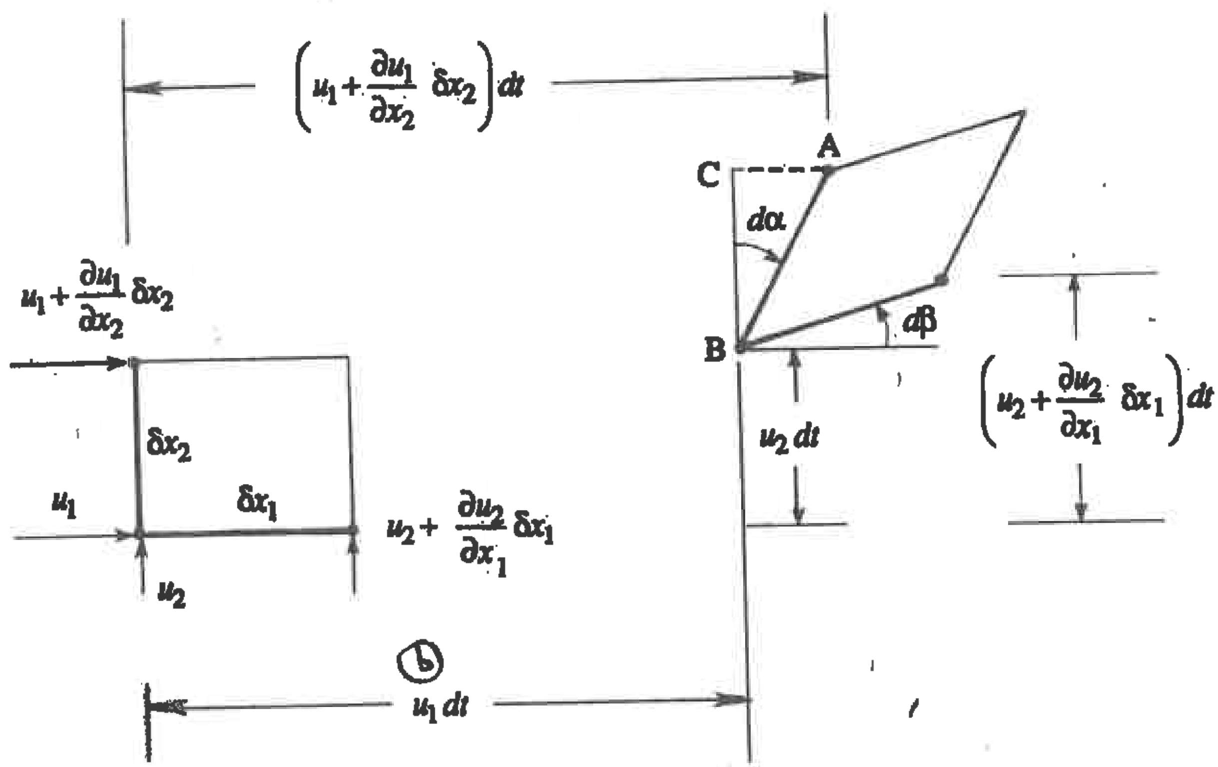
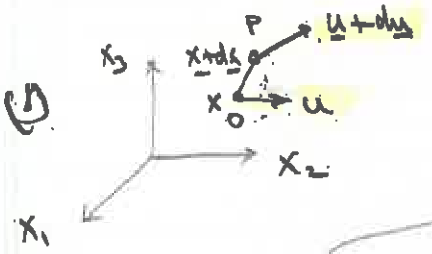


Figure 3.10 Deformation of a fluid element. Here, $d\alpha = CA/CB$; a similar expression represents $d\beta$.

Motion of nearby point:

(1)



relative velocity

$$\boxed{d\underline{u} = \frac{\partial u_i}{\partial x_j}\, dx_j} \qquad (3.19)$$

$$= \frac{\partial u_1}{\partial x_1} dx_1 + \frac{\partial u_2}{\partial x_2} dx_2 + \frac{\partial u_3}{\partial x_3} dx_3$$

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{symmetric}} + \underbrace{\frac{1}{2}\overbrace{\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}^{\text{rotation}}}_{\text{antisymmetric}}$$

angular velocity

(2)

$$\boxed{\frac{\partial u_i}{\partial x_j} = e_{ij} + \frac{1}{2} r_{ij}}$$

strain rate — rotation tensor

$\underline{\underline{r}}$ ← average rotation = rotation tensor

saw last time
average Rotation $\equiv \underline{\underline{r}}$
$= \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$

Any anti-symmetric tensor can be specified by a vorticity vector, $\underline{\omega}$, which is twice the angular velocity (because of the ½ multiplier above)

$$\underline{\underline{r}} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \qquad \underline{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

rotation tensor corresponding to vorticity vector ω

vorticity vector

(3)

$$\boxed{\underline{\omega} = \underline{\nabla} \times \underline{u}}$$

← particle velocity

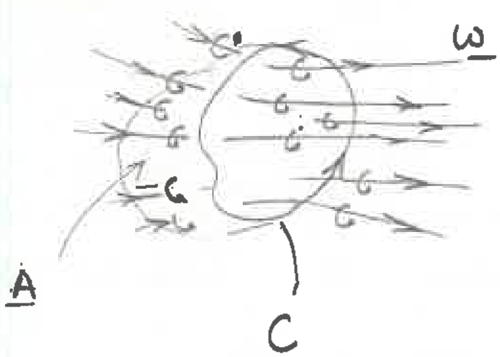
(4-3) Vorticity

$$\boxed{\underline{\omega} = \nabla \times \underline{u}}$$

Using Stokes' Theorem,

$$\text{circulation} \quad \Gamma = \oint_C \underline{u} \cdot d\underline{s} = \int_A \underbrace{(\nabla \times \underline{u})}_{=\text{vorticity vector}, \, \underline{\omega}} \cdot d\underline{A} = \int_A \underline{\omega} \cdot d\underline{A}$$

This says the circulation around an area is the total flux of vorticity through that area.



Flux of vorticity into area enclosed by the bounding surface = flux through surface containing C

which can be thought of as a planar cut.

(4-4) Circulation

$$\boxed{\Gamma \equiv \int_A \underset{\text{vorticity}}{\underline{\omega}} \cdot d\underline{A} = \oint_C \underline{u} \cdot d\underline{s}}$$

The circulation, Γ, is defined as the total vorticity flux through a surface of area A.

Lemmas

Curl #1

$$\underline{u} \times \underline{v} = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\hat{x}_1 (u_2 v_3 - u_3 v_2)$$
$$\hat{x}_2 (u_3 v_1 - u_1 v_3)$$
$$\hat{x}_3 (u_1 v_2 - u_2 v_1)$$

Curl #2

(4) $$(\underline{u} \times \underline{v})_k = \epsilon_{ijk} u_i v_j = \epsilon_{kij} u_i v_j$$

Alternating tensor or permutation symbol

$\epsilon_{ijk} = 0$ if any index equal

index can be moved 2 places w/ same value

$\epsilon_{ijk} = 1$

123 clockwise

$\epsilon_{ijk} = -1$

132 counter clockwise

Rotation tensor corresp. to vorticity vector w/

$$\underline{\underline{r}} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

Example

$$\underline{\omega} = \underline{\nabla} \times \underline{u}$$
$$\omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$
$$= \epsilon_{ijk} \partial_j u_k$$
$$= \epsilon_{ijk} u_{k,j}$$

Idle

comma denotes

Rotation Tensor + Vorticity vector

(5) $$\boxed{r_{ij} = -\epsilon_{ijk} \omega_k}$$

row col

$$r_{12} = -\omega_3$$
$$r_{13} = \omega_2$$
$$r_{21} = \omega_3$$
$$r_{23} = -\omega_1$$
$$r_{31} = -\omega_2$$
$$r_{32} = \omega_1$$

(6)

$$= \frac{\partial u_i}{\partial x_j} dx_j \quad \text{, or by (2)}$$

$$du = \left(\epsilon_{ij} + \frac{1}{2} r_{ij}\right) dx$$

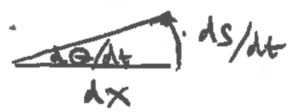
$$= e_{ij} dx_j - \frac{1}{2} \epsilon_{ijk} w_k dx_j$$

$$du_i = e_{ij} dx_j + \frac{1}{2} \epsilon_{ikj} w_k dx_j$$

swap once change sign

$w = \frac{d\theta}{dt}$ (c)

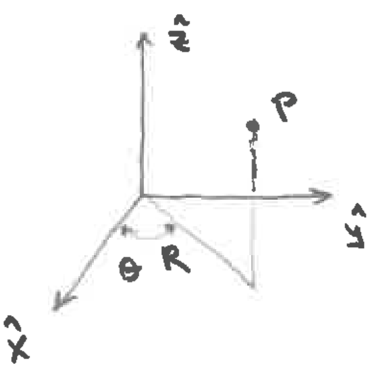
$$\boxed{\underset{\text{velocity}}{d\underline{u}} = \underline{\underline{e}} \cdot d\underline{x} + \frac{1}{2} \underbrace{\left(\overset{\text{vorticity}}{\underline{w}} \times d\underline{x}\right)}}$$



velocity at distance dx from a point due to rotation at angular velocity w

velocity at p due to deformation + velocity at p due to rotation at angular velocity $\underline{w}/2$

Cylindrical Coordinates



$\nabla \times u =$

$$\hat{r}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right) +$$

$$\hat{\theta}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) +$$

$$\Rightarrow \hat{z}\,\frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right)$$

Now consider some examples:

A. Solid body rotation

(See diagram)

$$\boxed{\begin{aligned} u_\theta &= \omega_0 r \\ u_r &= 0 \end{aligned}}$$

$$\underline{\omega} = \underline{\nabla} \times \underline{u}$$

$$\omega_z = \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right) \quad \to 0$$

$$= \frac{u_\theta}{r} + \frac{r}{r}\frac{\partial u_\theta}{\partial r}$$

$$= \frac{\omega_0 \cancel{r}}{\cancel{r}} + \frac{\partial \omega_0 r}{\partial r} = \overbrace{2\omega_0}^{\text{vorticity}}$$

circulation

$$\Gamma = \oint u \cdot ds = \int_0^{2\pi} u_\theta r \, d\theta = 2\pi r u_\theta$$

$$= 2\underbrace{\pi r^2}_{\text{area}} \omega_0$$

$$= \underbrace{2\omega_0}_{} \cdot \text{area}$$

Total vorticity flux

deformation is zero (solid body rotation)

Vorticity $= 2\omega_0$

all parts of body rotate about their centers!

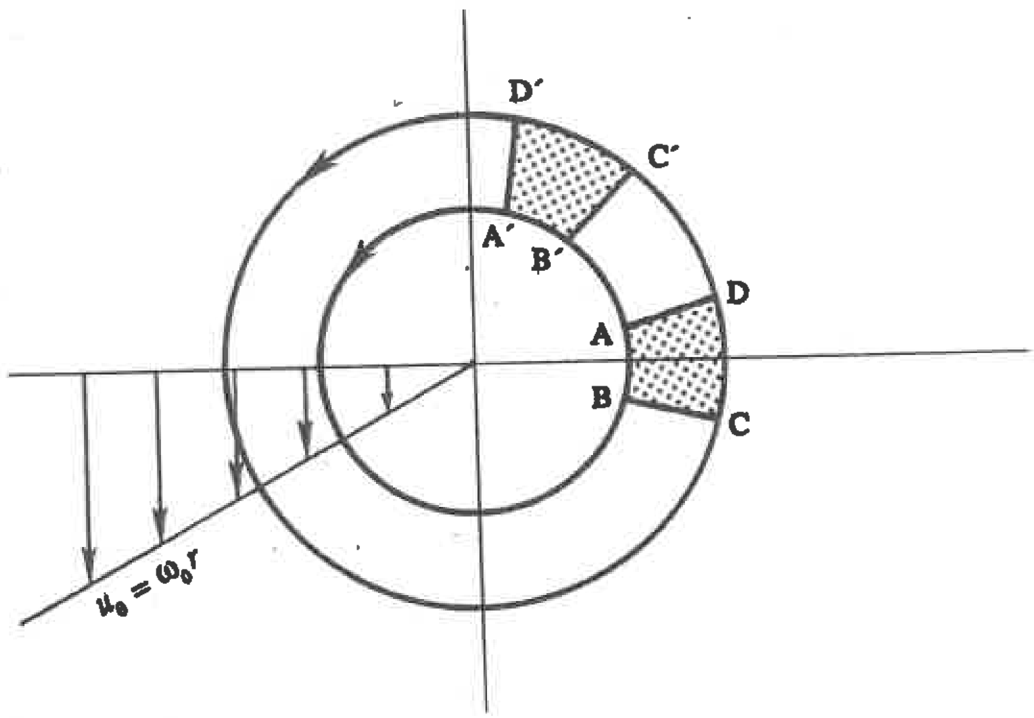


Figure 3.15 Solid-body rotation. Fluid elements are spinning about their own centers while they revolve around the origin. There is no deformation of the elements.

B. Irrotational Vortex

$$\boxed{\begin{aligned} u_\theta &= c/r \\ u_r &= 0 \end{aligned}}$$

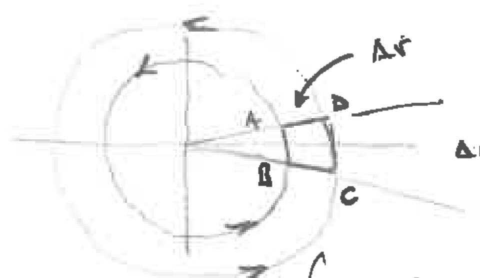
$$\omega = \nabla \times u$$

$$\omega_z = \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \cancel{\frac{\partial u_r}{\partial \theta}}^{\,0}\right)$$

$$= \frac{1}{r}\left(\frac{\partial}{\partial r}\left(\frac{rc}{r}\right)\right) = 0$$

Vorticity is zero at any point of flow

Easy to see this is the case



$\Gamma = \oint u \cdot ds$

Circulation around any point not containing the origin is zero. ∴ vorticity flux thru circuit is zero ∴ vorticity is zero

$$\oint u \cdot ds = - u_\theta(r_1)\, r_1\, d\theta + u_\theta(r_2)\, r_2\, d\theta$$

$\frac{c}{r_1} r_1\, d\theta$

$\frac{c}{r_2} r_2\, d\theta$

But circulation around any circuit containing the origin is $2\pi c$

$$\Gamma = \int_0^{2\pi} \frac{c}{r}(r\, d\theta) = 2\pi c$$

Consider Stokes Theorem :

$$\Gamma = \oint u \cdot ds = \int_A \underline{\omega} \cdot dA = 2\pi C$$

because Γ = constant, integral independent of A and can shrink A to near zero. The vorticity at origin must be infinite so that $\underline{\omega} \cdot \delta A$ = finite

∴ under irrotational flow — vorticity is zero everywhere except at origin where it is infinite!

This is a singularity! Complex variables — any contour enclosing singularity gets its contribution!

Mathematics comes from physics! [It makes a big difference if I can understand physically what is going on but mathematics make physics elegant.]

C. The Rankine Vortex

An irrotational vortex is a natural description of a Tornado, except tornadoes are not quite singularities. A nice "fix" is to "patch" together solid body rotation and an irrotational vortex :

Solid body core $\quad r \le R$

$$u_\theta = \omega_0 r$$

$$\Gamma = 2\omega_0 A$$

Irrotational extrapolate to ∞

$$u_\theta = C/r$$

$$\Gamma = 2\pi C$$

Chapter 3 problem set is excellent! not easy but excellent.

Remember calculus. Think!

7. Kronecker delta and Epsilon delta

Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Epsilon delta relationship

$$\underbrace{\epsilon_{ijk}\,\epsilon_{klm}}_{\text{summed}} = \overbrace{\delta_{il}\,\delta_{jm}}^{\text{first}} - \overbrace{\delta_{im}\,\delta_{jl}}^{\text{others}}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ i & j & i & j \end{matrix}$$

D. Double Contraction

$$\overset{S}{\underline{\underline{B}}} : \underline{\underline{A}} = \overset{S}{\underline{\underline{B}}} : \overset{S}{\underline{\underline{A}}}$$

$$\overset{S}{\underline{\underline{B}}} : \overset{A}{\underline{\underline{A}}} = 0$$

where superscript A = antisymmetric, S = symmetric

No proof in Book

Proof

A very frequently occurring operation is the doubly contracted product of a *symmetric* tensor τ and any tensor **B**. The doubly contracted product is defined as

$$P \equiv \tau_{ij} B_{ij} = \tau_{ij}(S_{ij} + A_{ij}),$$

where **S** and **A** are the symmetric and antisymmetric parts of **B**, given by

$$S_{ij} \equiv \tfrac{1}{2}(B_{ij} + B_{ji}) \quad \text{and} \quad A_{ij} \equiv \tfrac{1}{2}(B_{ij} - B_{ji}).$$

Then

$$\begin{aligned} P &= \tau_{ij} S_{ij} + \tau_{ij} A_{ij} && && (2.28) \\ &= \tau_{ij} S_{ji} - \tau_{ij} A_{ji} && \text{because } S_{ij} = S_{ji} \text{ and } A_{ij} = -A_{ji}, \\ &= \tau_{ji} S_{ji} - \tau_{ji} A_{ji} && \text{because } \tau_{ij} = \tau_{ji}, \\ &= \tau_{ij} S_{ij} - \tau_{ij} A_{ij} && \text{interchanging dummy indices.} && (2.29) \end{aligned}$$

Comparing the two forms of equations (2.28) and (2.29), we see that $\tau_{ij} A_{ij} = 0$, so that

$$\boxed{\tau_{ij} B_{ij} = \tfrac{1}{2}\tau_{ij}(B_{ij} + B_{ji}).}$$

The important rule we have proved is that the *doubly contracted product of a symmetric tensor τ with any tensor **B** equals τ times the symmetric part of **B***. In the process, we have also shown that the doubly contracted product of a symmetric tensor and an antisymmetric tensor is zero. This is analogous to the result that the definite integral over an even (symmetric) interval of the product of a symmetric and an antisymmetric function is zero.

Note more complex stream in Backup notebook

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